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NOTE

Closed-Form Expressions for Certain Induction Integrals Involving Jacobi and Chebyshev Polynomials

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1. INTRODUCTION

This paper gives closed-form expressions for certain integrals that appear in the numerical solution of Cauchy-singular integral equations by collocation methods. This class of integral equations is frequently associated with problems of potential theory involving finite strip boundaries [1, 2, 3]. Applications may be found, for example, in airfoil theory, elasticity, and hydrodynamics [2]. The formulas we derive effectively eliminate the numerical instabilities associated with standard recursion formulas.

The integrals considered are of the form $\int_{-1}^{1} w(x)\, p_n(x)\, dx/(x-z)$, where p_n denotes a Jacobi or Chebyshev polynomial, w a weighting function, and z a complex field coordinate. Typically, boundary values on the strip $(-1, 1)$ are expressed as series in p_n. The integrals then express the field induced at a point z by the boundary values. When there are multiple boundary strips, the integrals also represent mutual induction between boundaries; in that case, they must be considered during the boundary-value solution process.

A prototype system of boundary integral equations for a two-strip problem, for example, is

$$P.V. \int_{a_1}^{b_1} \frac{f_1(x_1)}{x-x_1}\, dx_1 + \int_{a_2}^{b_2} \frac{f_2(x_2)}{x-x_2}\, dx_2 = g_1(x), \qquad a_1 < x < b_1 \tag{1a}$$

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$$\int_{a_1}^{b_1} \frac{f_1(x_1)}{x - x_1}\,dx_1 + P.V.\int_{a_2}^{b_2} \frac{f_2(x_2)}{x - x_2}\,dx_2 = g_2(x), \qquad a_2 < x < b_2, \tag{1b}$$

where f_1, f_2 are unknown boundary values on the strips (a_1, b_1) and (a_2, b_2) lying on the x-axis, and g_1, g_2 are given functions, typically up-wash velocities in thin-airfoil theory. In a collocation scheme, the principal-value integrals are conveniently handled by expressing the unknown potentials in terms of Jacobi or Chebyshev polynomials, for example,

$$f(x_1) = \sqrt{\frac{1 + x_1}{1 - x_1}} \sum_n a_n P_n^{(-\frac{1}{2},\frac{1}{2})}(x_1), \tag{2}$$

where the range of x_1 is normalized to $(-1, 1)$, and the $P_n^{(-1/2,1/2)}$ are Jacobi polynomials. The square-root factor in this case allows an integrable trailing-edge singularity, in thin-airfoil theory parlance. The $P_n^{(-1/2,1/2)}$ are orthogonal on $(-1, 1)$ with respect to the square-root factor, and thus form a basis for a collocation scheme. Behavior of the surface values at the endpoints $-1, 1$ is related to the *index* of the solution in the theory of singular integral equations [1, 3]. In applications, its choice is dictated by physical considerations. The correct polynomial expansion follows from this choice. For example, a solution that is integrably singular at -1 and zero at $+1$ would involve the factor $\sqrt{(1-x)/(1+x)}$ and the polynomials $P_n^{(1/2,-1/2)}$.

The theory of such boundary collocation schemes is covered in [1]. In this note, we are concerned with the generation of field values, given the collocation coefficients. Thus the computation of fluid velocity from a surface vorticity distribution, for example, involves integrals of the form

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, \frac{P_n^{(-\frac{1}{2},\frac{1}{2})}(x)}{x - z}\,dx$$

for a vorticity representation of the form (2). The field point z (complex) may lie anywhere except on the strip $(-1, 1)$. It is seen that the non-singular mutual-induction integrals appearing in the boundary integral equations (1a), (1b) are also of this form.

We present closed-form expressions for induction integrals involving $P_n^{(1/2,-1/2)}$, $P_n^{(-1/2,1/2)}$ (Jacobi polynomials), and as a by-product, U_n, T_n (Chebyshev polynomials).

NOTATION AND RECURSION FORMULAS

We work with the normalized Jacobi polynomials or "airfoil polynomials" [1]

$$u_n(x) = \frac{n!}{\Gamma(n+1/2)} P_n^{(\frac{1}{2},-\frac{1}{2})}(x), \qquad v_n(x) = \frac{n!}{\Gamma(n+1/2)} P_n^{(-\frac{1}{2},\frac{1}{2})}(x) \tag{3}$$

which satisfy the orthonormality relations

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} u_n(x)\, u_m(x)\,dx = \delta_{mn}, \qquad \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} v_n(x) v_m(x)\,dx = \delta_{mn} \tag{4}$$

and the recursion relations

$$u_{n+1} = 2xu_n - u_{n-1}, \qquad v_{n+1} = 2xv_n - v_{n-1}. \tag{5}$$

The induction integrals are defined as

$$I_n(z) = \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \frac{u_n(x)}{x-z}\, dx, \qquad K_n(z) = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \frac{v_n(x)}{x-z}\, dx. \tag{6}$$

From the expressions

$$u_0 = v_0 = \frac{1}{\sqrt{\pi}}, \qquad u_1 = \frac{1}{\sqrt{\pi}}(2x+1), \qquad v_1 = \frac{1}{\sqrt{\pi}}(2x-1)$$

and the recursion relations (5), it is easy to derive the following recursion formulas for the integrals:

$$I_0(z) = \sqrt{\pi} - i\sqrt{\pi}\sqrt{\frac{1-z}{1+z}}, \qquad I_1(z) = 2\sqrt{\pi} + (2z+1)I_0(z) \tag{7a}$$

$$I_{n+1}(z) = 2zI_n(z) - I_{n-1}(z) \tag{7b}$$

and similarly for $K_n(z)$. In (7a), the branch of the square root is such that its result lies in the upper half-plane.

CLOSED-FORM EXPRESSIONS

The recursion scheme (7) is seriously corrupted by numerical noise for $n > 12$ or so and $|z| > 2$ (Fig. 1). We now show that it is possible to derive closed-form expressions for $I_n(z)$ and $K_n(z)$ that are free of such problems.

For $I_n(z)$, we consider the Chebyshev polynomials $U_n(x)$ of the second kind [4] and the associated integrals

$$\hat{I}_n(z) = \int_{-1}^{1} \sqrt{1-x^2}\, \frac{U_n(x)}{x-z}\, dx. \tag{8}$$

Using known relationships [4] between the $U_n(x)$ and the $P_n^{(1/2,-1/2)}$, we can derive

$$u_n(x) = \frac{1}{\sqrt{\pi}} U_{2n}\left(\sqrt{\frac{1+x}{2}}\right).$$

Substituting into (6) and making use of the fact that $U_{2n}(x)$ is even, we find that

$$I_n(z) = \sqrt{\frac{1}{2\pi(1+z)}} \left[\hat{I}_{2n}\left(\sqrt{\frac{1+z}{2}}\right) - \hat{I}_{2n}\left(-\sqrt{\frac{1+z}{2}}\right)\right]. \tag{9}$$

In order to evaluate $\hat{I}_n$, we make use of the following representation of $U_n(x)$ [4],

$$U_n(x) = \frac{1}{2\pi i} \oint_C \frac{s^{-n-1}\, ds}{1-2xs+s^2}, \tag{10}$$

where the path C must enclose the origin but exclude the zeros of $(1-2xs+s^2)$. Since $-1 < x < 1$ in our case, these zeros lie on the unit circle. Thus any circuit enclosing the

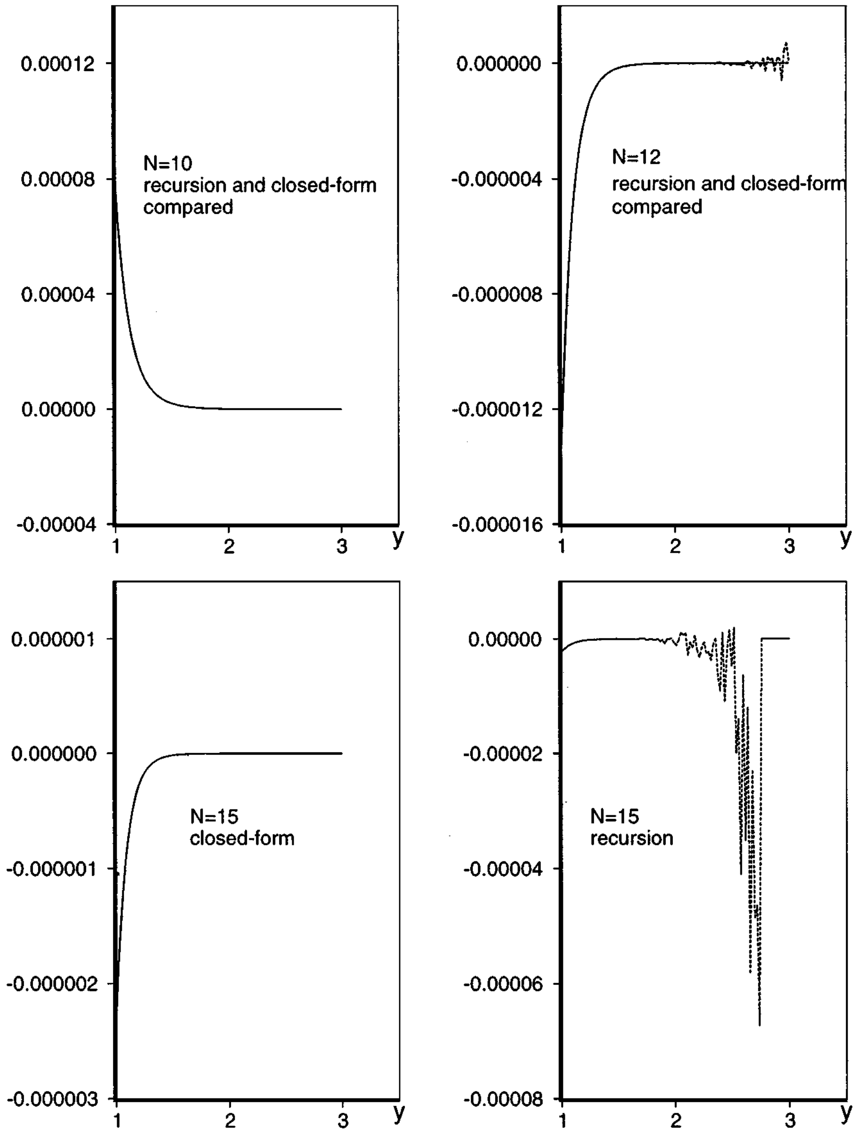

FIG. 1. Recursion (dashed lines) vs closed-form (solid lines) calculations for the integral $\mathrm{Re}[K_n(z)]$, for $z = 0 + iy$ and $n = 10$, 12, 15.

origin and lying wholly inside the unit circle is admissible. We will presently impose one more restriction on C.

Inserting (10) into (8) and interchanging the order of integration, we obtain

$$\hat{I}_n(z) = \frac{1}{2\pi i} \oint_C \frac{\hat{I}_0(z) + \pi s}{1 - 2zs + s^2} \frac{ds}{s^{n+1}}, \tag{11a}$$

where

$$\hat{I}_0(z) = -\pi z - i\pi(1+z)\sqrt{\frac{1-z}{1+z}} \tag{11b}$$

and the branch of the square root is such that its result lies in the upper half-plane.

Let $z_{1,2}$ denote the zeros of $(1 - 2zs + s^2)$. It is impossible for either zero to lie at the origin. Thus the contour C can be made small enough that both $z_{1,2}$ lie outside it. In that case the only singularity of the integrand in (11a) is the one at the origin due to $1/s^{n+1}$. Furthermore, the expansion

$$\frac{1}{1-2zs+s^2} = \frac{1}{z_1 z_2} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left(\frac{s}{z_1}\right)^p \left(\frac{s}{z_2}\right)^q \tag{12}$$

is permissible. Using (12) and the relations $z_1 z_2 = 1, z_1 + z_2 = 2z$, we can compute the residue. Working through the algebra, we arrive at

$$\hat{I}_n(z) = \frac{\hat{I}_0(z)}{[\hat{I}_0(z)/\pi + 2z]^n}. \tag{13}$$

Expressions (9), (11b), and (13) combine to give a closed-form expression for $I_n(z)$. Note that (13) gives an expression for the induction integral (8), involving Chebyshev polynomials of the second kind.

A similar development leads to an expression for $K_n(z)$, which involves the normalized Jacobi polynomials υ_n. Using relationships given in [4], we can write

$$\upsilon_n(x) = \sqrt{\frac{2/\pi}{1+x}}\, T_{2n+1}\left(\sqrt{\frac{1+x}{2}}\right),$$

where T_n denotes a Chebyshev polynomial of the first kind. Define the associated integrals

$$\hat{K}_n(z) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{T_n(x)}{x-z}\, dx. \tag{14}$$

Substituting the above expression for υ_n into the definition (6) of $K_n(z)$, and using the fact that T_{2n+1} is an odd function, we find that

$$K_n(z) = \frac{1}{2\sqrt{\pi}} \left[\hat{K}_{2n+1}\left(\sqrt{\frac{1+z}{2}}\right) + \hat{K}_{2n+1}\left(-\sqrt{\frac{1+z}{2}}\right)\right]. \tag{15}$$

Next, we use the contour integral representation [4] for T_n,

$$T_n(x) = \frac{1}{4\pi i} \oint_C \frac{s^{-n-1}(1-s^2)}{1-2xs+s^2}\, ds, \tag{16}$$

where C must enclose the origin and exclude both zeros of the denominator, which lie on the unit circle. Proceeding as we did for $I_n(z)$, we obtain

$$\hat{K}_n(z) = \frac{1}{4\pi i} \oint_C \frac{(1-s^2)\hat{K}_0(z) + 2\pi s}{1-2zs+s^2} \frac{ds}{s^{n+1}}, \tag{17}$$

where

$$\hat{K}_0(z) = \frac{i\pi}{2}\left[\sqrt{\frac{1+z}{1-z}} - \sqrt{\frac{1-z}{1+z}}\right] \tag{18}$$

and the range of the square root function is restricted to the upper half-plane. Restricting C as before and using the expansion (12) to calculate the residue of (17), we arrive at

$$\hat{K}_n(z) = \frac{\hat{K}_0(z)}{2[z - \pi/\hat{K}_0(z)]^n} + \frac{2\pi z + (2z^2 - 1)\hat{K}_0(z)}{2[z - \pi/\hat{K}_0(z)]^{n-2}}. \tag{19}$$

Equations (19), (18), and (15) constitute the result for $K_n(z)$, while (19) and (18) give us the value of the Chebyshev induction integral (14).

NUMERICAL COMPARISONS

Figure 1 gives a comparison between the recursion formula (7) and the closed-form expressions given above. It is seen that the recursion formula rapidly loses stability for moderate values of z, while the closed-form expressions remain robust. The instability of the recursion formula is greater for higher-order polynomials. For large $|z|$, both $I_n(z)$ and $K_n(z)$ have asymptotic expansions in inverse powers of z; these are easily derived by expanding the $\frac{1}{x-z}$ factor in (6) in powers of x, which may then be expressed in terms of u_n or v_n as appropriate. It is found that the closed form expressions remain robust in the asymptotic regime as well, roughly $|z| > 3$. Thus the formulas are suitable for use over the entire range of n and z, provided z does not lie on the real-axis segment $(-1, 1)$.

"Higher-order" induction integrals, involving powers $1/(x - z)^m$, are easily obtained by repeated differentiation of the formulas obtained here. We note also the comments of a reviewer, who pointed out that some of these formulas may be derived in a relatively straightforward manner from results given in [5].

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